Name: ____________________ **MATH 158-03 F16** **Review for Exam 4**

Exam 4 will be similar to WebAssign, written homework, and examples done in class from the following sections.

16.1 Vector Fields

16.2 Line Integrals

16.3 The Fundamental Theorem for Line Integrals

16.4 Green's Theorem

16.5 Curl and Divergence

16.6 Parametric Surfaces and Their Areas

16.7 Surface Integrals

Note that on WebAssign under Personal Study Plan, you can practice quizzes and chapter quizzes. *This is highly recommended to do!*

Here is a brief list of topics but this list may not include all topics that may appear on the Exam. Note that some of these may be recast in two dimensions.

16.1 Vector Fields

Let E be a set in $\mathbb{R}^3$. A **vector field on** $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point (x, y, z) in E a two-dimensional vector $\mathbf{F}(x, y, z)$. We write $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ or $\mathbf{F} = \langle P, Q, R \rangle$.

A **gradient vector field** is $\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}$.

$\mathbf{F}$ is a **conservative vector field** if there is a function f such that $\mathbf{F} = \nabla f$, and f is called the **potential function** for $\mathbf{F}$.

16.2 Line Integrals

For a smooth curve C traversed once as t increases from a to b, the **line integral of** f **along** C is

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\,\Delta s_i &= \int_C f(x, y, z)\,ds \\ &= \int_a^b f(x(t), y(t), z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt \\ &= \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\,dt\end{aligned}$$

If $\rho(x, y, z)$ is the linear density of a thing wire in the shape of a smooth curve C, then

$$m = \int_C \rho(x, y, z)\,ds, \quad \bar{x} = \frac{1}{m}\int_C x\rho(x, y, z)\,ds, \quad \bar{y} = \frac{1}{m}\int_C y\rho(x, y, z)\,ds, \quad \bar{z} = \frac{1}{m}\int_C z\rho(x, y, z)\,ds$$

The **line integrals of f along C with respect to x, y, and z** are:

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\,\Delta x_i = \int_C f(x,y,z)\,dx = \int_a^b f(x(t),y(t),z(t))x'(t)\,dt$$
$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\,\Delta y_i = \int_C f(x,y,z)\,dy = \int_a^b f(x(t),y(t),z(t))y'(t)\,dt$$
$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\,\Delta z_i = \int_C f(x,y,z)\,dy = \int_a^b f(x(t),y(t),z(t))z'(t)\,dt$$

We write $\int_C P(x,y,z)\,dx + \int_C Q(x,y,z)\,dy + \int_C R(x,y,z) = \int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$.

Popular **parameterizations** of a curve C

- A **line** from $\mathbf{r}_0$ to $\mathbf{r}_1$ is given by solving $\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1$ for $\langle x(t), y(t)\rangle$, where t goes from 0 to 1.
- A **circle** with center (h,k) and radius a is given by solving $\frac{x-h}{a} = \cos t$, $\frac{y-k}{b} = \sin t$ for $\langle x(t), y(t)\rangle$, where t goes from 0 to 2π.

For **orientation** of a smooth curve C, we have

$$\int_{-C} f(x,y,z)\,dx = -\int_C f(x,y,z)\,dx, \qquad \int_{-C} f(x,y,z)\,dy = -\int_C f(x,y,z)\,dy,$$
$$\int_{-C} f(x,y,z)\,dz = -\int_C f(x,y,z)\,dz, \qquad \int_{-C} f(x,y,z)\,ds = \int_C f(x,y,z)\,ds.$$

The line integral of a continuous vector field $\mathbf{F} = \langle P, Q, R\rangle$ along C is

$$W = \int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz$$

where W is the work done by the force field F in moving a particle along C. Note that $\mathbf{F}(\mathbf{r}(t)) = \mathbf{F}(x(t), y(t), z(t))$ and $d\mathbf{r} = \mathbf{r}'(t)\,dt = \langle dx, dy, dz\rangle$.

16.3 The Fundamental Theorem for Line Integrals

The **Fundamental Theorem of Line Integrals** (FTOLI): If C is a smooth curve C traversed by $\mathbf{r}(t)$ for $a \le t \le b$ and f is differentiable with a continuous gradient vector ∇f on C, then

$$\int_C \nabla f\cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

We say $\int_C \mathbf{F}\cdot d\mathbf{r}$ is **independent of path** if for any two paths C_1, C_2 in D with the same initial and terminal points, $\int_{C_1}\mathbf{F}\cdot d\mathbf{r} = \int_{C_2}\mathbf{F}\cdot d\mathbf{r}$.

Th 16.3.3: $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$ for every closed path C in D.

Th 16.3.4: For a continuous vector field $\mathbf{F}$ on an open connected region D, if $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D.

Th 16.3.5-6: For a vector field $\mathbf{F} = \langle P, Q\rangle$ on an open simply-connected region D where P and Q have continuous first-order partials, $P_y = Q_x$ on D if and only if $\mathbf{F}$ is conservative.

16.4 Green's Theorem, 16.5 Curl and Divergence

If $\mathbf{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ is a three-dimensional vector field, then the:

- **curl** of $\mathbf{F}$ is curl $\mathbf{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle$. Note that curl $\mathbf{F} = \nabla \times \mathbf{F}$.
- **divergence** of $\mathbf{F}$ is div $\mathbf{F} = P_x + Q_y + R_z$. Note that div $\mathbf{F} = \nabla \cdot \mathbf{F}$.

If $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ is a two-dimensional vector field, then curl $\mathbf{F} = Q_x - P_y$ and div $\mathbf{F} = P_x + Q_y$.

Th 16.5.3: If $f(x,y,z)$ is a function with continuous second-order partials, then curl $(\nabla f) = \mathbf{0}$.

Th 16.5.4 (Conservative Vector Field Test): If $\mathbf{F}$ is a vector field defined on $\mathbb{R}^3$ who component functions have continuous partials, then curl $\mathbf{F} = \mathbf{0}$ if and only if $\mathbf{F}$ is conservative.

Green's Th: Let C be a positively oriented, piecewise-smooth, simple closed curve in $\mathbb{R}^2$ and let D be the region bounded by C. If $\mathbf{F} = \langle P, Q\rangle$ where P and Q have continuous partials on an open region that contains D, then $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA$.

The curl of $\mathbf{F}$ measures the tendency of the "fluid" flow to produce rotation and $\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA = \iint_D \text{curl } \mathbf{F}\,dA$, where $\mathbf{T}$ is the unit tangent.

The divergence of $\mathbf{F}$ measures the amount of outward flux of a vector field in a very small area (or volume) around a point and $\int_C \mathbf{F}\cdot\mathbf{N}\,ds = \int_C P\,dy - Q\,dx = \iint_R (P_x + Q_y)\,dA = \iint_R \text{div } \mathbf{F}\,dA$, where $\mathbf{N}$ is the unit outward normal.

The scalar projection of $\mathbf{F}$ onto $\mathbf{T}$ is the signed magnitude of the vector projection and is given by $\text{comp}_{\mathbf{T}}\mathbf{F} = \frac{\mathbf{F}\cdot\mathbf{T}}{|\mathbf{T}|} = \mathbf{F}\cdot\mathbf{T}$. Integrating this tangential component of a vector field is used to compute work and measure circulation along C.

16.6 Parametric Surfaces and Their Areas

The vector-valued function $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ for (u,v) in $D \subset \mathbb{R}^2$ defines a **parametric surface** S in the xyz-plane. S is smooth if $\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$.

Grid curves in the xyz-plane are found by holding $v = v_0$ or $u = u_0$, where v_0, u_0 are some constants.

Popular **parameterizations** of a surface S:

- Sphere of radius a: $\mathbf{r}(\phi,\theta) = \langle a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi\rangle$
- Graph of a function $z = f(x,y)$: $\mathbf{r}(u,v) = \langle u, v, f(u,v)\rangle$
- Graph of a function $f(\theta, r)$: $\mathbf{r}(u,v) = \langle v\cos u, v\sin u, f(u,v)\rangle$
- Plane containing P_0, $\mathbf{a}$, and $\mathbf{b}$: $\mathbf{r}(u,v) = \mathbf{r}_0 + u\mathbf{a} + u\mathbf{b}$
- Surface of Revolution, Rotate $y = g(z)$ about the z-axis: $\mathbf{r}(u,v) = \langle g(v)\cos u, g(v)\sin u, v\rangle$
- Circular Cylinder: $\mathbf{r}(u,v) = \langle \cos u, \sin u, v\rangle$
- Cone: $\mathbf{r}(u,v) = \langle v\cos u, v\sin u, v\rangle$
- Paraboloid: $\mathbf{r}(u,v) = \langle \sqrt{v}\cos u, \sqrt{v}\sin u, v\rangle$

If a smooth parametric surface S is given by $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ for (u,v) in D, then the **surface area** of S is: $A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\,dA = \iint_D dS$.

16.7 Surface Integrals

Surface integral of f over the surface S: $\sum_{m,n\to\infty}^{m}\sum^{n} f(P_{ij}^*)\,\Delta S_{ij} = \iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\,dA.$

For a thin sheet in the shape of S with density ρ,

$$m = \iint_S \rho(x,y,z)\,dS, \quad \bar{x} = \frac{1}{m}\iint_S x\rho(x,y,z)\,dS, \qquad \bar{y} = \frac{1}{m}\iint_S y\rho(x,y,z)\,dS, \qquad \bar{z} = \frac{1}{m}\iint_S z\rho(x,y,z)\,dS.$$

If $\mathbf{F}$ in continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{N}$, then the **surface integral of F over S** is

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{N}\,dS = \iint_D \mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,dA = \text{ flux of } \mathbf{F} \text{ across } S.$$

Extra – Will Not Be Included

Stokes' Th: Let S ban an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F}\cdot d\mathbf{S}.$$

Recall that $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$.

Divergence Th: Let E be a simple solid region and let $S = \partial E$, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partials on an open region that contains E. Then

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \text{div } \mathbf{F}\,dV.$$

Recall that $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{N}\,dS$.