Name: ______________________ **MATH 158-03 F16** **Review for Exam 3**

Exam 3 will be similar to WebAssign, written homework, and examples done in class from the following sections.

15.1 Double Integrals over Rectangles

15.2 Double Integrals over General Regions

15.3 Double Integrals in Polar Coordinates

15.4 Applications of Double Integrals

15.5 Surface Area

15.6 Triple Integrals

15.7 Triple Integrals in Cylindrical Coordinates

15.8 Triple Integrals in Spherical Coordinates

15.9 Change of Variables in Multiple Integrals

Note that on WebAssign under Personal Study Plan, you can practice quizzes and chapter quizzes. *This is highly recommended to do!*

Note that the geometry of the domain you are integrating on determines the type of coordinate system you use. It is essential to include sketches in your work so that you can "see" the solids as well as their projections on a plane. Here is a brief list of topics but this list may not include all topics that may appear on the Exam.

15.1 Double Integrals over Rectangles

The **double integral** of f over R is $\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$, if this limit exists.

We can **estimate** the value of the integral by taking sample points (x_{ij}^*, y_{ij}^*) to be the midpoint of $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, or the upper right corner of each R_{ij}. Note that for $z = f(x,y)$, we can get the z-values from an explicit formula, a table, or a contour plot.

If $f(x,y) \geq 0$, the **volume** above R and below the surface $z = f(x,y)$ is $V = \iint_R f(x,y)\,dA$.

Fubini's Theorem: If f is continuous on $R = [a,b] \times [c,d]$, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

Multiplicatively Separable: If $R = [a,b] \times [c,d]$, then $\iint_R g(x)h(y)\,dA = \int_a^b g(x)\,dx \int_c^d h(y)\,dy$.

The **average value** of f on R is

$$f_{ave} = \frac{1}{A(R)} \iint_R f(x,y)\,dx\,dy$$

where $A(R)$ is the area of the region R.

15.2 Double Integrals over General Regions

Type I: If $D = \{(x,y)\,|\,a \le x \le b,\, g_1(x) \le y \le g_2(x)\}$, then $\iint\limits_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$.

Type II: If $D = \{(x,y)\,|\,h_1(y) \le x \le h_2(y),\, c \le y \le d\}$, then $\iint\limits_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$.

Properties:

- The **length** of $[a,b]$ is: $\ell([a,b]) = \int_a^b 1\,dx$
- The **area** of a region D is: $A(D) = \iint\limits_D 1\,dA$
- The **volume** of a solid E is: $V(E) = \iiint\limits_E 1\,dV$
- If $m \le f(x,y) \le M$ for all (x,y) in D, then $mA(D) \le \iint\limits_D f(x,y)\,dA \le MA(D)$

15.3 Double Integrals in Polar Coordinates

The rectangular coordinates (x,y) have corresponding **polar coordinates** (r,θ) where $x = r\cos\theta$ and $y = r\sin\theta$. Also, $x^2 + y^2 = r^2$.

Double Integral in Polar Coordinates: $\iint\limits_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$ where $R = \{(r,\theta)\,|\,\alpha \le \theta \le \beta,\, a \le r \le b\}$.

If $D = \{(r,\theta)\,|\,\alpha \le \theta \le \beta,\, h_1(\theta) \le r \le h_2(\theta)\}$, then $\iint\limits_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

15.4 Applications of Double Integrals

Suppose a lamina (or thin plate) has density $\rho(x,y)$ in D.

- Mass of the lamina: $m = \iint\limits_D \rho(x,y)\,dA$.
- Moment of the lamina about the x-axis: $M_x = \iint\limits_D y\,\rho(x,y)\,dA$
- Moment of the lamina about the y-axis: $M_y = \iint\limits_D x\,\rho(x,y)\,dA$
- Center of mass of the lamina: $(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right)$

15.5 Surface Area

The **area of the surface** $z = f(x,y)$ on D, where z_x and z_y are continuous: $A(S) = \iint\limits_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA$

15.6 Triple Integrals

Fubini's Theorem: If f is continuous on the box $B = [a,b] \times [c,d] \times [r,s]$, then

$$\iiint\limits_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz.$$

If $E = \{(x,y,z)\,|\,(x,y) \in D,\; u_1(x,y) \le z \le u_2(x,y)\}$, then $\iiint\limits_E f(x,y,z)\,dV = \iint\limits_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA$.

Examples: If $E = \{(x,y,z)\,|\,a \le x \le b,\; g_1(x) \le y \le g_2(x),\; u_1(x,y) \le z \le u_2(x,y)\}$, then

$$\iiint\limits_E f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx.$$

If $E = \{(x,y,z)\,|\,h_1(y) \le x \le h_2(y),\; c \le y \le d,\; u_1(x,y) \le z \le u_2(x,y)\}$, then

$$\iiint\limits_E f(x,y,z)\,dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dx\,dy.$$

...and so on...but be sure to graph the region E in $\mathbb{R}^3$ and its projection D in $\mathbb{R}^2$ to determine the limits of integration.

15.7 Triple Integrals in Cylindrical Coordinates

From rectangular coordinates (x,y,z), we have the corresponding **cylindrical coordinates** (r,θ,z) where

$$x = r\cos\theta, \quad y = r\sin\theta, \quad \text{and} \quad z = z.$$

This system uses polar coordinates in the xy-plane, and the directed distance from the xy-plane to the point.

If $E = \{(x,y,z)\,|\,(x,y) \in D,\; u_1(x,y) \le z \le u_2(x,y)\}$ and D can be written nicely in polar coordinates, then

$$\iiint\limits_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta.$$

15.8 Triple Integrals in Spherical Coordinates

From rectangular coordinates $P(x,y,z)$, we have the corresponding **spherical coordinates** (ρ,θ,ϕ) where

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad \text{and} \quad z = \rho\cos\phi.$$

Note that $\rho = |OP| \ge 0$ and $0 \le \phi \le \pi$, where O is the origin and ϕ is the angle between the positive z-axis and $|OP|$.

If the spherical wedge $E = \{(\phi,\theta,\phi)\,|\,a \le \rho \le b,\; \alpha \le \theta \le \beta,\; c \le \phi \le d\}$, then

$$\iiint\limits_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi.$$

15.9 Change of Variables in Multiple Integrals

The **Jacobian** of the transformation T given by $x = g(u,v)$ and $y = h(u,v)$ is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

To **change the variables**, (1) find $x = x(u,v)$ and $y = (u,v)$; (2) transform R into S; and (3) find the Jacobian of the transformation. Then,

$$\iint\limits_R f(x,y)\,dA = \iint\limits_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv.$$