Name: ________________________ **MATH 158-03 F16** **Review for Exam 2**

Exam 2 will be similar to WebAssign, written homework, and examples done in class from the following sections.

14.1 Functions of Several Variables

14.2 Limits and Continuity

14.3 Partial Derivatives

14.4 Tangent Planes and Linear Approximations

14.5 The Chain Rule

14.6 Directional Derivatives and the Gradient Vector

14.7 Maximum and Minimum Values

14.8 Lagrange Multipliers

Here is a brief list of topics but this list may not include all topics that may appear on the Exam. Note that many of the following are extendable from $z = f(x, y)$ to $w = f(x, y, z)$.

14.1 Functions of Several Variables

For a function of two variables, the **graph** of f is the set of points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and $(x, y) \in \text{Dom}(f)$.

The **level curves** of $z = f(x, y)$ are the curves with equations $f(x, y) = k$, where k is a constant (or a fixed z-value). We graph these, labeling each k-value, in the xy-plane.

14.2 Limits and Continuity

The **limit** of $f(x, y)$ as (x, y) approaches (a, b) is L and we write $\lim\limits_{(x,y)\to(a,b)} f(x, y) = L$.

If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path P_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path P_2 with $L_1 \neq L_2$, then $\lim\limits_{(x,y)\to(a,b)} f(x, y)$ does **not** exist.

Strategy to Evaluate Limits:

- If (a, b) is in the domain of the expression, then substitute.
- Try paths $(0, y)$, $(x, 0)$, (x, mx), (x, x^2), (x, x^n), etc. If the results are different, the limit does not exists. If the results are the same, maybe the limit does exist so go to next bullet.
- Rewrite the expression via algebra or in polar coordinates, and then evaluate the limit.
- Use the Squeeze Theorem to squeeze the expression between two expressions that bound the given expression, and have the same limits.

A function is **continuous** at (a, b) if $\lim\limits_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

14.3 Partial Derivatives

The **partial derivatives** of the function $f(x, y) = z$ are

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h} \qquad f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

Notation: $f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$ (Similarly for f_y.)

Clairaut's Theorem: If f_{xy} and f_{yx} are continuous on a disk that contains (a, b), then $f_{xy}(a, b) = f_{yx}(a, b)$.

To show that a function is a **solution** to a partial differential equation, find the derivatives, substitute, and show that it satisfies the equation.

14.4 Tangent Planes and Linear Approximations

Tangent plane to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0) is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Linear approximation or **linearization** of f at (a, b): $L(x, y) = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$
When (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$, the:

- **Increment** of z is $\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$
- **Differential** of z is $dz = f_x(a, b)dx + f_y(a, b)dy$

14.5 The Chain Rule

If $z = f(x, y)$ where $x = g(t)$ and $y = h(t)$, then $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$.

If $z = f(x, y)$ where $x = g(s, t)$ and $y = h(s, t)$, then $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} + \frac{\partial y}{\partial s}$ and $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$.

For **implicit differentiation**, first rewrite the equation to define an F. Then, for:

- $F(x, y) = 0$, $\frac{dy}{dx} = -\frac{F_x}{F_y}$.
- $F(x, y, z) = 0$, $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$ and $\frac{\partial z}{\partial x} = -\frac{F_y}{F_z}$.

14.6 Directional Derivatives and the Gradient Vector

The **gradient** of f is $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$.

The **directional derivative** of a function f in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is $D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b = \nabla f(x, y) \cdot \mathbf{u}$. If $\mathbf{u}$ is a unit vector given by angle θ, then $\mathbf{u} = \langle cos\theta, \sin\theta \rangle$.

The **maximum value of the directional derivative** is $D_{\mathbf{u}} f(x, y)$ is $|\nabla f(x, y)|$ and occurs when $\mathbf{u}$ is the same direction as the gradient vector $\nabla f(x, y)$.

First rewrite the equation to find $F(x, y, z) = k$. The **tangent plane** is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

The **normal line** at (x_0, y_0, z_0) has **symmetric equations**

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0}$$

and **parametric equations**

$$x = x_0 + F_x(x_0, y_0, z_0)t, \quad y = y_0 + F_y(x_0, y_0, z_0)t, \quad z = z_0 + F_z(x_0, y_0, z_0)t.$$

14.7 Maximum and Minimum Values

If $\nabla f(a, b) = \mathbf{0}$, then (a, b) is a **critical point** of f.

Types of Points/Values

- A function $z = f(x, y)$ has a **local maximum** at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b). The **local maximum value** is $f(a, b)$.
- A function $z = f(x, y)$ has a **local minimum** at (a, b) if $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b). The **local minimum value** is $f(a, b)$.
- A **saddle point** is a critical point, but the function has neither a maximum nor a minimum value at this point.

Note that "near" means in a disk with small radius and center (a, b).

Second Derivative Test: Let $D = f_{xx}(a, b)f_{yy}(a, b) - (f_{xy}(a, b))^2$.

- If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum of f.
- If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum of f.
- If $D < 0$, then (a, b) is a saddle point of f.
- If $D = 0$, then the test in inconclusive (the point could be anything).

Extreme Value Theorem+: If f is continuous on a closed, bounded set S, then f attains an absolute maximum and minimum values in S. The absolute maximum and minimum values of f will occur at the critical points of f in S, or on the boundary of S.

14.8 Lagrange Multipliers

To find the extreme values of $f(x, y, z)$ subject to the constraints $g(x, y, z) = k$ and $h(x, y, z) = c$:

- Find all values x, y, z, λ, and μ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z), \qquad g(x, y, z) = k, \qquad h(x, y, z) = c.$$

- Evaluate f at all of the points found. The largest is the maximum value of f and the smallest is the minimum value of f, subject to the constraints g and h. (If there is only one constraint g, then $\mu = 0$.)