Lecture Notes on Series

Series: a sum of a sequence

Notation: $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$

Definition 1. A **partial sum** is the sum up to the n^{th} term and is $s_n = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i$. If $s_n \to s$ where s is a real numbers, then we say that the series **converges** to s. That is, $\sum_{n=1}^{\infty} a_n = s$.

Definition 2. A **geometric series** can be written as $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$

Can we find this sum? Let's investigate the partial sums.

$$s_n = a + ar + \cdots + ar^{n-1}$$
$$rs_n = ar + ar^2 + \cdots + ar^n$$

Taking the difference of these equations we obtain $s_n - rs_n = a - ar^n$. Solving for s_n we get

$$s_n = \frac{a(1-r^n)}{1-r}$$

If $-1 < r < 1$, then $r^n \to 0$ and we want $r \neq 1$. In this case, $s_n \to \frac{a}{1-r}$ where a is the "seed" of the series and r is the ratio of successive terms, $\frac{a_{n+1}}{a_n}$. In general,

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad \text{if} \quad -1 < r < 1$$

Definition 3. We say that the series $\sum a_n$ is

- **absolutely convergent** if $\sum a_n$ and $\sum |a_n|$ both converge
- **conditionally convergent** if $\sum a_n$ converges but $\sum |a_n|$ diverges

Tests for Convergence/Divergence

Geometric Series: For $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$, if $|r| < 1$, the sum is convergent and equals $\frac{a}{1-r}$.
If $1 \leq |r|$, the sum is divergent.

p-Series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Test for Divergence: If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

The Integral Test: Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then,

$$\int_1^{\infty} f(x)\,dx \text{ converges } \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges} \qquad \textbf{OR} \qquad \int_1^{\infty} f(x)\,dx \text{ diverges } \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges}$$

Note: the corresponding integral behaves the same as the sum with respect to divergence/convergence.

The Comparison Test: Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with $a_n, b_n > 0$.

(i) $\sum_{n=1}^{\infty} b_n$ converges and $a_n \leq b_n \quad \Rightarrow \quad \sum_{n=1}^{\infty} a_n$ converges (note: $0 \leq \sum a_n \leq \sum b_n$)

(ii) $\sum_{n=1}^{\infty} b_n$ diverges and $b_n \leq a_n \quad \Rightarrow \quad \sum_{n=1}^{\infty} a_n$ diverges (note: $0 \leq \sum b_n \leq \sum a_n$)

Limit Comparison Test: Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with $a_n, b_n > 0$. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ and $0 < c < \infty$, then either both series converge or both diverge.

Note: for n large, $a_n \approx cb_n$ and so both sums behave the same with respect to divergence/convergence.

Alternating Series Test: $\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$ where $b_n > 0$ converges if

(i) $b_{n+1} \leq b_n$ (or b_n is decreasing) **AND** (ii) $\lim_{n\to\infty} b_n = 0$

Convergence of Absolute Values Implies Convergence: If $\sum_{n=1}^{\infty} |a_n|$ converges, then so does $\sum_{n=1}^{\infty} a_n$.

The Ratio Test: For $\sum_{n=1}^{\infty} a_n$, we evaluate the limit of the ratios $|a_{n+1} \div a_n|$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \begin{cases} L < 1, & \text{the series is absolutely convergent} \\ L > 1, & \text{the series is divergent} \\ L = 1, & \text{you get nothing} \end{cases}$$

The Root Test: For $\sum_{n=1}^{\infty} a_n$, we evaluate the limit of the roots $\sqrt[n]{|a_n|}$.

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \begin{cases} L < 1, & \text{the series is absolutely convergent} \\ L > 1, & \text{the series is divergent} \\ L = 1, & \text{you get nothing} \end{cases}$$